

POSITIVE CURVATURE, PARTIAL VANISHING THEOREMS AND COARSE INDICES

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1. INTRODUCTION

Let M be a complete Riemannian manifold and let D be a generalized Dirac operator acting on sections of a Clifford bundle S over M . It is well-known (see for example [1]) that there is a Weitzenböck formula

$$D^2 = \nabla^* \nabla + R,$$

where R is a certain self-adjoint endomorphism of S constructed out of the curvature. (For example, in the classical case of the Dirac operator associated to a spin-structure, R is pointwise multiplication by $\frac{1}{4}$ times the scalar curvature [4]).

The author's coarse index theory associates to D an index that lies in the K -theory of the “translation C^* -algebra” $C^*(M)$. As in the classical case, the index vanishes if the curvature operator is uniformly bounded below by a positive constant. In [7, Proposition 3.11] this statement is generalized as follows. Suppose that there is a subset $Z \subseteq M$, such that for some constant $a > 0$ one has $R_x \geq a^2 I$ (as self-adjoint endomorphisms of S_x) for all $x \notin Z$ — we will then say that the operator R is *uniformly positive outside* Z . Then the index of D lies in the image of the map

$$K_*(C^*(Z)) \rightarrow K_*(C^*(X)),$$

where Z is considered as a metric subspace of X . In particular, if the curvature is uniformly positive outside a compact set Z (so that $C^*(Z)$ is the compact operators), one recovers the result of Gromov and Lawson [1, Chapter 3] that D has an index in the ordinary Fredholm sense.

I included only the briefest sketch of a proof of this proposition in [7]. This note is a response to several requests for more detail, and also mentions a couple of applications of the idea.

2. THE MAIN RESULT

Let Z be a subset of a proper metric space X and let H be an ample X -module (i.e. a Hilbert space which is a “sufficiently large” module over

$C_0(X)$, assumed fixed — I refer to [2] for terminology. The module action is denoted by $\rho: C_0(X) \rightarrow \mathfrak{B}(H)$. The C^* -algebra $C^*(X)$ (or $C^*(X; H)$ if it is important to keep track of the particular Hilbert space) is then defined to be the norm closure of the controlled, locally compact operators on H , where we recall that a *controlled* (a.k.a. *finite propagation*) operator T has the property that there is a constant r for which

$$d(\text{Supp } \varphi, \text{Supp } \psi) > r \implies \rho(\varphi)T\rho(\psi) = 0$$

for all $\varphi, \psi \in C_c(X)$. A controlled operator T is *supported near* Z if there is another constant r' for which

$$d(\text{Supp } \varphi, Z) > r' \implies \rho(\varphi)T = 0 = T\rho(\varphi).$$

The norm closure of the set of controlled, locally compact operators supported near Z is an ideal in $C^*(X)$, which we denote¹ by $C^*(Z \subseteq X)$. It is easy to see [3] that the K -theory of $C^*(Z \subseteq X)$ is the same as that of $C^*(Z)$, if Z is considered as a metric space in its own right.

Now we recall the relation of these concepts to index theory. Suppose that X is actually a complete Riemannian manifold and that S is a Clifford bundle, and let $H = L^2(X; S)$ in forming the algebras above. The algebra $D^*(X)$ is defined to be the norm closure of the controlled, *pseudolocal* operators on H : it is a unital C^* -algebra, and $C^*(X)$ is an ideal in it. The following key analytic lemma [2, Chapter 10] can be proved by the finite propagation speed method:

Lemma 2.1. *Let X be a complete Riemannian manifold, as above, and let S be a Clifford bundle over it. Let D denote the Dirac operator of S , considered as an unbounded, self-adjoint operator on $H = L^2(X; S)$. If f is a bounded continuous function on \mathbb{R} that has finite limits at $\pm\infty$, then $f(D) \in D^*(X; H)$. If f tends to zero at $\pm\infty$, then $f(D) \in C^*(X; H)$. \square*

A *normalizing function* $\chi: \mathbb{R} \rightarrow [-1, 1]$ is, by definition, a continuous, odd function that tends to ± 1 at $\pm\infty$. Given such a function χ , it follows from the preceding lemma that $\chi(D) \in D^*(X)$ and $\chi(D)^2 - 1 \in C^*(X)$. Moreover, if χ_1 and χ_2 are two normalizing functions, then it similarly follows that $\chi_1(D) - \chi_2(D) \in C^*(X)$. Thus the equivalence class of $\chi(D)$ gives a well-defined self-adjoint involution in $D^*(X)/C^*(X)$, defining an element $[\chi(D)] \in K_{j+1}(D^*(X)/C^*(X))$ (j is determined by the grading of the operator — it is equal to the parity of $\dim X$). Now we have

Definition 2.2. With the notation of Lemma 2.1, the *coarse index* of D is

$$\text{Index}(D) = \partial[\chi(D)] \in K_j(C^*(X)),$$

¹It is denoted $C_X^*(Z)$ in [7], but the other notation now seems better to me.

where $\partial: K_{j+1}(D^*(X)/C^*(X)) \rightarrow K_j(C^*(X))$ is the boundary map in the long exact sequence of C^* -algebra K -theory.

Now let $Z \subseteq X$ as above. The algebra $C^*(Z \subseteq X)$ is an ideal in $D^*(X)$ (not just in $C^*(X)$). To prove our result we will need to sharpen Lemma 2.1 as follows:

Lemma 2.3. *Let notation be as in Lemma 2.1. Suppose that the curvature operator $R = R_D$ that appears in the Weitzenbock formula for D ,*

$$D^2 = \nabla^* \nabla + R_D,$$

is uniformly positive outside Z , say $R_x \geq a^2 I$ for $x \notin Z$. Then for any $f \in C_c(-a, a)$ we have $f(D) \in C^(Z \subseteq X)$.*

Suppose that this lemma has been proved. Then choose a normalizing function χ such that $\chi^2 - 1$ is supported in $(-a, a)$. According to Lemma 2.3, the equivalence class of $\chi(D)$ is a (well-defined) self-adjoint involution in $D^*(X)/C^*(Z \subseteq X)$. Following the construction above, we obtain a *localized index*

$$\text{Index}_Z(D) \in K_j(C^*(Z \subseteq X))$$

which maps to the previously defined $\text{Index}(D)$ under the K -theory map induced by the inclusion $C^*(Z \subseteq X) \rightarrow C^*(X)$. The existence of this localized index is the precise content of [7, Proposition 3.11]; it implies the version of the result stated in the introduction. To state it precisely:

Theorem 2.4. *Let M be a complete Riemannian and let D be a Dirac-type operator whose associated curvature endomorphism R_D is uniformly positive outside a subset Z of M . Then the construction above defines a localized coarse index*

$$\text{Index}_Z(D) \in K_j(C^*(Z \subseteq X))$$

which maps to the coarse index $\text{Index}(D) \in K_j(C^(M))$ under the K -theory map induced by the inclusion $C^*(Z \subseteq X) \rightarrow C^*(X)$. (Here j is the parity of $\dim M$.)*

The rest of this section will give the proof of Lemma 2.3. In order to use the finite propagation speed method, we consider first the properties of functions f that have compactly supported Fourier transforms.

Lemma 2.5. *With notation as in Lemma 2.3, suppose that $f \in \mathcal{S}(\mathbb{R})$ is an even function and has Fourier transform \hat{f} supported in $(-r, r)$. Let $\varphi \in C_0(X)$ have support disjoint from a $2r$ -neighborhood of Z . Then*

$$\|f(D)\rho(\varphi)\| \leq \|\varphi\| \sup\{|f(\lambda)| : |\lambda| \geq a\}$$

and the same estimate applies to $\rho(\varphi)f(D)$.

Proof. We use the Fourier cosine formula

$$f(D) = \frac{1}{\pi} \int_0^r \hat{f}(t) \cos(tD) dt,$$

remembering that $\hat{f}(t)$ vanishes for $t > r$. Now let $U_n = \{x \in X : d(x, Z) > nr\}$, for $n = 1, 2$, and consider the unbounded, symmetric operator which is equal to D^2 with domain $C_c^\infty(U_1)$. This operator is bounded below by $a^2 I$ and therefore it has a Friedrichs extension on the Hilbert space $L^2(U_1; S)$ which is also bounded below (with the same bound) and which we shall denote by E .

A standard finite propagation speed argument shows that if s is smooth and compactly supported in U_2 then

$$\cos(tD)s = \cos(t\sqrt{E})s, \quad \text{for } 0 \leq t \leq r.$$

In particular, $\cos(tD)\rho(\varphi) = \cos(t\sqrt{E})\rho(\varphi)$ for these values of t . Via the Fourier integral above, this implies that $f(D)M_\varphi = f(\sqrt{E})M_\varphi$. But since the spectrum of \sqrt{E} is bounded below by a ,

$$|f(\sqrt{E})| \leq \sup\{|f(\lambda)| : |\lambda| \geq a\},$$

and this gives the desired estimate. \square

There is a version of Lemma 2.5 without the evenness hypothesis.

Lemma 2.6. *With notation as above, suppose that $f \in \mathcal{S}(\mathbb{R})$ has Fourier transform \hat{f} supported in $(-r, r)$. Let $\varphi \in C_0(X)$ have support disjoint from a $4r$ -neighborhood of Z . Then*

$$\|f(D)\rho(\varphi)\| \leq 2\|\varphi\| \sup\{|f(\lambda)| : |\lambda| \geq a\}$$

and the same estimate applies to $\rho(\varphi)f(D)$.

Proof. If f is even, this is a consequence of Lemma 2.5. If f is odd, use the C^* -identity to write

$$\|f(D)\rho(\varphi)\|^2 \leq \|\rho(\bar{\varphi})\| \| |f|^2(D)\rho(\varphi) \|.$$

The function $g = |f|^2$ is even, belongs to $\mathcal{S}(R)$ and has Fourier transform supported in $(-2r, 2r)$. Thus, applying Lemma 2.5 to the function g ,

$$\| |f|^2(D)\rho(\varphi) \| \leq \|\varphi\| \sup\{|f(\lambda)|^2 : |\lambda| \geq a\}$$

and so we obtain (on taking the square root)

$$\|f(D)\rho(\varphi)\| \leq \|\varphi\| \sup\{|f(\lambda)| : |\lambda| \geq a\},$$

which gives the desired result for odd f . The general result is obtained by writing f as a sum of even and odd components (this decomposition accounts for the extra factor of 2 in the statement of Lemma 2.6). \square

Using this, let us complete the proof of Lemma 2.3. Let f be as in that lemma, and let $\varepsilon > 0$ be given. There exists a smooth function g with compactly supported Fourier transform such that $\sup\{|g(\lambda) - f(\lambda)| : \lambda \in \mathbb{R}\} < \varepsilon$. In particular, $|g(\lambda)| < \varepsilon$ for $|\lambda| > a$. Let r be such that $\text{Supp}(\hat{g}) \subseteq (-r, r)$ and let $\psi: X \rightarrow [0, 1]$ be a continuous function equal to 1 on a $4r$ -neighborhood of Z and vanishing off a $5r$ -neighborhood of Z . Write

$$\begin{aligned} f(D) &= \rho(\psi)g(D)\rho(\psi) + \\ &\quad + \rho(1 - \psi)g(D)\rho(\psi) + g(D)\rho(1 - \psi) + (f(D) - g(D)). \end{aligned}$$

The first term is a locally compact operator supported near Z , the second and third terms have norm bounded by 2ε by lemma 2.6, and the fourth term has norm bounded by ε by the spectral theorem. Thus, $f(D)$ lies within 5ε of a locally compact operator supported near Z . Since ε is arbitrary, $f(D) \in C^*(Z \subseteq X)$, as was to be shown.

3. VANISHING RESULTS

As a consequence of the discussion above, if the curvature operator R is uniformly positive outside Z , and if the K -theory map $K_*(C^*(Z)) \rightarrow K_*(C^*(X))$ is zero, then the index $\text{Index}(D) \in K_*(C^*(X))$ must vanish. The usual vanishing theorem establishes this result when $Z = \emptyset$ (i.e., when we have uniformly positive curvature on the whole of M), so we can regard these sort of results as a generalization where one allows a “small amount” of non-positive curvature.

For example, we have

Proposition 3.1. *Let M be a complete connected noncompact Riemannian manifold, and let D be a Dirac-type operator whose associated curvature R is uniformly positive outside a compact set. Then $\text{Index}(D) = 0$.*

Proof. Let K be a compact set outside which the curvature is uniformly positive, and let Z be the union of K and a geodesic ray from one of its points to infinity. The index of D then lies in the image of $K_*(C^*(Z)) \rightarrow K_*(C^*(X))$ by the discussion above. But Z is coarsely equivalent to \mathbb{R}^+ , so $K_*(C^*(Z)) = 0$. \square

For another example, imagine that we are in the situation of the “partitioned manifold index theorem” of [5]. So, let M be a non-compact manifold that is partitioned by a compact hypersurface N , which (say) is spin and of non zero \hat{A} -genus, into two pieces M^+ and M^- .

Proposition 3.2. *A partitioned manifold as described above admits no complete metric that has uniformly positive scalar curvature on just one of the partition components (M^+ or M^-).*

Proof. Suppose M has such a metric. Using the distance from N , construct a proper, coarse map $g: M \rightarrow \mathbb{R}$ that induces the given partition. By definition, the partitioned manifold index is

$$g_*(\text{Index } D) \in K_1(C^*(|\mathbb{R}|)) = \mathbb{Z},$$

and the index theorem of [5] equates this to the $\widehat{\mathcal{A}}$ -genus of N . Now suppose that M has positive scalar curvature over M^+ . Then by our main result, the coarse index factors through $K_1(C^*(M^- \subseteq M))$. But considering the commutative diagram

$$\begin{array}{ccc} K_1(C^*(M^- \subseteq M)) & \longrightarrow & K_1(C^*(M)) \\ \downarrow g_* & & \downarrow g_* \\ K_1(C^*(\mathbb{R}^- \subseteq \mathbb{R})) & \longrightarrow & K_1(C^*(\mathbb{R})) \end{array}$$

and noting that the bottom left-hand group is zero, we see that the coarse index vanishes. \square

4. THE RELATIVE INDEX THEOREM

The key technical result of [1, Chapter 4] is a relative index theorem which may be expressed as follows.

Suppose that M_1 and M_2 are complete Riemannian manifolds equipped with generalized Dirac operators D_1 and D_2 respectively, acting on (graded) Clifford bundles S_1 and S_2 . Suppose further that these items *agree near infinity*: in other words, that there exist compact sets $Z_i \subseteq M_i$ an isometry $h: M_1 \setminus Z_1 \rightarrow M_2 \setminus Z_2$ that is covered by a bundle isomorphism from S_1 to S_2 , and that this isomorphism conjugates D_1 to D_2 .

In these circumstances one can define a *relative topological index* $\text{Index}_r(D_1, D_2) \in \mathbb{Z}$. There are several ways to define this quantity. For instance, one can compactify each of the M_i identically outside Z_i (thus obtaining *compact* manifolds \widetilde{M}_i with elliptic operators \widetilde{D}_i) and then take the difference of the ordinary Fredholm indices, $\text{Index}(\widetilde{D}_1) - \text{Index}(\widetilde{D}_2)$, to define the relative index. Alternatively, one can take the Chern-Weil forms \mathfrak{a}_i that are the representatives of the indices of D_i according to the local index theorem, and “integrate their difference” over $M_1 \cup M_2$: specifically, note that h^* takes \mathfrak{a}_2 to \mathfrak{a}_1 , so that if we let \mathfrak{a} be any smooth form on M_2 , supported outside Z_2 and agreeing with \mathfrak{a}_2 near infinity, then the difference

$$\int_{M_1} (\mathfrak{a}_1 - h^*\mathfrak{a}) - \int_{M_2} (\mathfrak{a}_2 - \mathfrak{a})$$

is well-defined (the integrands are compactly supported) and independent of the choice of \mathfrak{a} , and may be taken as the definition of the “integral of the difference of Chern-Weil forms”. The equality of these two definitions of

relative index is essentially Proposition 4.6 of [1]: it shows both that the first definition is independent of the choice of compactification, and that the second definition yields an integer.

Remark 4.1. Either definition implies that the relative index $\text{Index}_r(D_1, D_2)$ depends only on the geometry of M_1 and M_2 (and the associated operators) in a neighborhood of the “regions of disagreement” Z_1 and Z_2 . This stability property of the relative index is the basis for several calculations in [1].

Now suppose further that D_1 and D_2 have uniformly positive Weitzenböck curvature operators at infinity. Then D_1 and D_2 , individually, are Fredholm operators, by Theorem 3.2 of [1] (a special case of our Theorem 2.4). The relative index theorem then states

Proposition 4.2. [1, Theorem 4.18] *In the circumstances described above one has*

$$\text{Index}(D_1) - \text{Index}(D_2) = \text{Index}_r(D_1, D_2).$$

We are going to generalize this result by allowing the “regions of disagreement” Z_i to be non-compact. The first thing that we need to do is to *define* the relative index in this case. The following discussion, which is based on the ideas of [6], leads up to the generalized definition of the relative index, Definition 4.5.

Let M_1 and M_2 be complete Riemannian manifolds (as above) and let D_1 and D_2 be generalized Dirac operators. Suppose that M_1 and M_2 are equipped with coarse maps q_1 and q_2 to a *control space* X (a proper metric space), and that Z is a subset of X . Put $Z_i = q_i^{-1}(Z) \subseteq M_i$ for $i = 1, 2$. Suppose that there is a diffeomorphism $h: M_1 \setminus Z_1 \rightarrow M_2 \setminus Z_2$ which is covered by an isomorphism of Clifford bundles and Dirac operators and which is *compatible with the control maps* in the sense that $q_1 = q_2 \circ f$.

From these data one can define a relative index in $K_j(C^*(Z))$. Let H_i be the Hilbert space $L^2(M_i; S_i)$ and regard each H_i as an X -module via the control map q_i . In this way we obtain translation algebras $C^*(X; H_i)$, $i = 1, 2$, each of which contains an ideal $C^*(Z \subseteq X; H_i)$ corresponding to Z . The isometry h between the M_i outside Z_i passes to a unitary isomorphism V between the $L^2(M_i \setminus Z_i; S_i)$, and it is easy to see that conjugation by this unitary induces an isomorphism of quotient C^* -algebras

$$\Phi: C^*(M_1; H_1)/C^*(Z_1 \subseteq M_1; H_1) \rightarrow C^*(M_2; H_2)/C^*(Z_2 \subseteq M_2; H_2).$$

Lemma 4.3. *Let notation be as above and let $f \in C_0(\mathbb{R})$. Then*

$$\Phi[f(D_1)] = [f(D_2)],$$

in the quotient algebra $C^(M_2; H_2)/C^*(Z_2 \subseteq M_2; H_2)$.*

There is also a “ D^* -version” of this discussion. Namely, following the forthcoming PhD thesis of Paul Siegel [9] we can define ideals $D^*(Z_i \subseteq M_i; H_i)$ as the closure of the finite propagation, pseudolocal² operators that are supported near Z_i and are locally compact on $M_i \setminus Z_i$. Once again, conjugation by U induces an isomorphism of quotient C^* -algebras

$$\Psi: D^*(M_1; H_1)/D^*(Z_1 \subseteq M_1; H_1) \rightarrow D^*(M_2; H_2)/D^*(Z_2 \subseteq M_2; H_2).$$

Lemma 4.4. *Let notation be as above and let χ be a normalizing function. Then*

$$\Psi[\chi(D_1)] = [\chi(D_2)],$$

in the quotient algebra $D^(M_2; H_2)/D^*(Z_2 \subseteq M_2; H_2)$.*

Proof. The proofs of both Lemmas 4.3 and 4.4 rely on the finite propagation speed method. First we give the proof for 4.3. Suppose that $f \in \mathcal{S}(\mathbb{R})$ and has Fourier transform \hat{f} supported in $(-r, r)$. As usual, we write

$$f(D) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) e^{itD} dt$$

and use the fact that, for a Dirac-type operator D , e^{itD} has propagation $|t|$. Let $\psi_i: M_i \rightarrow [0, 1]$ be a smooth function equal to 1 on an r -neighborhood of Z_i and vanishing off a $2r$ -neighborhood of Z_i , and such that $\psi_1 = \psi_2 \circ h$ on $M_1 \setminus Z_1$. Write

$$f(D_i) = f(D_i)\rho(\psi_i) + f(D_i)(1 - \rho(\psi_i)).$$

Since $f(D_i)$ has propagation r , the first term belongs to $C^*(Z_i \subseteq H_i)$. By finite propagation speed we have

$$V^* e^{itD_1} \rho(1 - \psi_1) V = e^{itD_2} \rho(1 - \psi_2) \quad \text{for } |t| < r.$$

Consequently,

$$V^* f(D_1)(1 - \rho(\psi_1))V = f(D_2)(1 - \rho(\psi_2))$$

and the proof is complete for f having compactly supported Fourier transform. The general result follows, since such f are norm-dense in $C_0(\mathbb{R})$.

The proof of Lemma 4.4 follows a similar pattern, where the Fourier transform $\hat{\chi}$ must now be understood as a distribution with a mild singularity at 0. The only additional argument that is needed is to show that

$$(4.1) \quad (V\chi(D_1)V^* - \chi(D_2))\rho(\varphi)$$

is compact for $\varphi \in C_0(M_2 \setminus Z_2)$. Suppose in fact that φ is compactly supported. Then there is a constant $r > 0$ such that $d(Z_2, \text{Supp}(\varphi)) > r$ and, if we should choose the normalizing function χ to have Fourier

²“Finite propagation” is defined with respect to the control space X via the control maps q_i ; “pseudolocal” is defined with respect to the ambient manifold M_i .

transform supported in $(-r, r)$, then finite propagation speed shows that the displayed quantity in 4.1 is not just compact — it is actually *zero*! The general case follows from this particular one, since any two normalizing functions differ by some $g \in C_0(\mathbb{R})$, and we already know that for such g , the *individual* terms $g(D_1)$ and $g(D_2)$ are locally compact. \square

Now let π_i denote the quotient map $C^*(M_i) \rightarrow C^*(M_i)/C^*(Z_i \subseteq M_i)$ or $D^*(M_i) \rightarrow D^*(M_i)/D^*(Z_i \subseteq M_i)$ as appropriate. Let us define A to be the pull-back C^* -algebra

$$A = \{(T_1, T_2) \in C^*(M_1; H_1) \oplus C^*(M_2; H_2) : \Phi(\pi_1(T_1)) = \pi_2(T_2)\}.$$

Similarly define B to be the pull-back C^* -algebra

$$B = \{(T_1, T_2) \in D^*(M_1; H_1) \oplus D^*(M_2; H_2) : \Phi(\pi_1(T_1)) = \pi_2(T_2)\}.$$

Then A is an ideal in B . Let D denote the Dirac operator on the disjoint union $M_1 \sqcup M_2$. Lemmas 4.3 and 4.4 above show that for a normalizing function χ , the operator $\chi(D)$ is an element of B , and that for a function $f \in C_0(\mathbb{R})$, the operator $f(D)$ is an element of the ideal A . Consequently there is defined an index of D

$$(4.2) \quad \text{Index}_Z(D) \in K_j(A).$$

The group $K_j(A)$ can be decomposed as a direct sum. In fact, let $U: H_1 \rightarrow H_2$ be a *covering isometry* for the identity map [2, Definition 6.3.9] that agrees on $L^2(M_1 \setminus Z_1)$ with the isomorphism $L^2(M_1 \setminus Z_1) \rightarrow L^2(M_2 \setminus Z_2)$ induced by h . (The hypothesis that h boundedly commutes with the control maps assures the existence of such an isometry.) Then there is a split short exact sequence

$$(4.3) \quad 0 \rightarrow C^*(Z_1 \subseteq M_1) \rightarrow A \rightarrow C^*(M_2) \rightarrow 0,$$

where the first map is $a \mapsto (a, 0)$, the second is $(a_1, a_2) \mapsto a_2$, and the splitting maps a to (U^*aU, a) . From this split short exact sequence we obtain a direct sum decomposition

$$K_j(A) = K_j(C^*(Z_1 \subseteq M_1)) \oplus K_j(C^*(M_2)).$$

Definition 4.5. The *relative index* of the above data is the component in $K_j(C^*(Z_1 \subseteq M_1)) = K_j(C^*(Z))$ of $\text{Index}_Z(D) \in K_j(A)$. We denote it by $\text{Index}_r(D_1, D_2)$.

The generalization of Gromov-Lawson's relative index theorem is then

Theorem 4.6. *Let (M_i, D_i, q_i) be a set of relative-index data over (X, Z) , with the notation described above. Suppose that the operators D_i have uniformly positive Weitzenböck curvature operators outside Z_i . Then each*

D_i has a localized coarse index in $K_j(C^*(Z))$, by Theorem 2.4, and the identity

$$\text{Index}_Z(D_1) - \text{Index}_Z(D_2) = \text{Index}_r(D_1, D_2)$$

holds in $K_j(C^*(Z))$.

(The case considered by Gromov and Lawson can be recovered by taking $X = \mathbb{R}^+$, $Z = \{0\}$.)

Proof. Let A be the pull-back algebra that we introduced in our definition of the relative index (so that A consists of pairs (T_1, T_2) , $T_i \in C^*(M_i)$, that “agree away from Z ”.) Let J be the ideal in A that consists of pairs (T_1, T_2) where each T_i belongs to $C^*(Z_i \subseteq M_i)$; in fact, J is simply the direct sum $C^*(Z_1 \subseteq M_1) \oplus C^*(Z_2 \subseteq M_2)$. Let D denote the Dirac operator on $M_1 \sqcup M_2$.

Because of the positive curvature away from Z it follows from Lemma 2.3 that, for $f \in C_0(\mathbb{R})$, $f(D)$ belongs to the ideal J . Thus, in this case, the index $\text{Index}_Z(D)$ defined in Equation 4.2 in fact belongs to $K_j(J) = K_j(C^*(Z)) \oplus K_j(C^*(Z))$, and it is apparent from the definitions that, in terms of this direct sum decomposition,

$$\text{Index}_Z(D) = (\text{Index}_Z(D_1), \text{Index}_Z(D_2)).$$

The definition of the relative index tells us to take the component of $\text{Index}_Z(D)$ in $K_j(C^*(Z))$ in the direct sum decomposition coming from the split short exact sequence 4.3. Restricted to J , this sequence takes the form

$$0 \rightarrow C^*(Z) \rightarrow C^*(Z) \oplus C^*(Z) \rightarrow C^*(Z) \rightarrow 0,$$

where the first map is inclusion on the first factor, the second is projection on the second factor, and the splitting used is $a \mapsto (a, a)$. Using this splitting, one finds that the relevant component of $\text{Index}_Z(D) = (\text{Index}_Z(D_1), \text{Index}_Z(D_2))$ is $\text{Index}_Z(D_1) - \text{Index}_Z(D_2)$, as required. \square

As we observed above, it is an important feature of the Gromov-Lawson relative index that it depends only on the geometry of a neighborhood of the “region of disagreement”. The corresponding result is also true in our more general context, and is a key to the applications of the relative index concept in [7].

Proposition 4.7. [7, Theorem 3.12] *The relative index of Definition 4.5 depends only on the geometry of a metric neighborhood of Z_1 and Z_2 and the operators thereon.*

Notice that this statement is independent of any positive-curvature hypotheses.

Proof. This follows from the results of [8]. In that paper, it is shown that to a set of relative index data (as described in this section), one may associate a *relative homology class* that lies in the K -homology group $K_*(Z)$. Moreover, comparison of the definitions shows that our coarse relative index is simply the image of this relative homology class under the coarse assembly map

$$A : K_*(Z) \rightarrow K_*(C^*(Z)).$$

The result is therefore a consequence of Proposition 4.8 of [8], which states that in fact the *relative homology class* of a set of relative index data depends only on the geometry in a neighborhood of the region of disagreement. \square

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